ResearchIntoPractice MATHEMATICS

Solving Word Problems: Developing Quantitative Reasoning

By Dr. Randall Charles

Problem solving has been the focus of a substantial number of research studies over the past thirty years. It is well beyond the scope of this paper to even attempt to summarize this body of research. Those interested in significantly broader reviews of research related to problem solving should see Schoenfeld (1985), Charles (1987), Charles & Silver (1988), and Lesh & Zawojewski (2007). This paper focuses on one area of research that has been of great interest to mathematics educators: solving mathematics "word problems." Some relevant research and implications for teaching are discussed in this paper.

Setting the Issue

The recently released Common Core State Standards for Mathematics consists of two sets of standards: the Standards for Mathematical Content and the Standards for Mathematical Practice. This latter set of standards, which describes the practices, processes and dispositions that teachers should look to develop in their students, highlights the continued importance that the mathematics community places on helping students become proficient in solving problems and reasoning mathematically.

The importance of helping students develop the abilities and skills related to solving problems cannot be understated. They are key foundational abilities and skills that students will draw on throughout their school years, and in their professional careers.

There are many types of mathematics problems that students regularly encounter in the school mathematics curriculum. (See Charles & Lester, 1982, for a classification of mathematics problems.) This paper focuses on a particular type of problem that many teachers refer to as "word problems." Some break this type of problem into "one-step word problems" and "multiple-step word problems." Charles and Lester (1982) call this type of problem a "translation problem." Word problems have been chosen as the focus of this paper for two reasons. First, because they are the most common type of problem-solving task found on assessments; they are likely to be prominent, if not dominant in the common assessments currently under development by the two consortia, the Partnership for Assessment of Readiness for College and Careers (PARCC) and Smarter Balanced Assessment Consortium (SBAC). And second, because the abilities and skills related to solving word problems are an important foundation for success in algebra.

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The issue with word problems is that too many students continue to be unsuccessful at solving them! Teachers still report that developing students' abilities to solve word problems is one of their most difficult and frustrating challenges. Students continue to have anxiety about solving problems, and they know that practice alone does not help them improve.

Here is a rather formal statement of what constitutes a word problem. This formal statement will be helpful when discussing implications for teaching later in this paper.

A **mathematics word problem** is a real-world situation in which mathematical quantities are given, values of one or more quantities are known, while the values of one or more quantities are unknown. The relationships between or among quantities are described, a question is implied or stated asking one to find the value of one or more unknown quantities, and one or more of the four operations – addition, subtraction, multiplication, and division – are to be used to find the value of the unknown quantity or quantities and answer the question. The problem might contain extraneous data, and it might contain one or more "hidden questions"—sub-problems that need to be solved in order to provide the final answer to the problem. The answer to the question might be numerical (e.g., "The car costs $23,000.") or not (e.g., "He has enough money to buy the car.").

Two widely taught strategies for problem solving are the key words approach and the problem-solving steps approach. For many elementary teachers, these strategies provide a logical and manageable, if not formulaic, structure to teaching students how to solve word problems. However, if problem solving continues to be difficult for so many students, one can only conclude that these common teaching strategies need to be challenged. An analysis of these two approaches can uncover their limitations.

Key Words: A "key words" approach teaches students to use a particular operation whenever they encounter a certain English word or phrase in a word problem. For example, students are instructed to use addition whenever the question in a word problem includes "in all."

A body of research is not needed to show the limited value of the key word approach. Quite often on state or national assessments, students encounter word problems where a key word approach is either not applicable or misleading. Problems may contain no words that might be connected to a particular operation or they may contain "misleading" key words (e.g., "in all" is in the question but addition is not the needed operation). The reason problems containing no key words or misleading ones are on assessments is not to set students up for failure. Rather, the problems look to measure how well students are able to identify the quantities in a problem, understand the relationship between quantities, and choose the operation(s) needed to find the solution. They do not look to measure how well students can apply a formulaic process that requires little thinking or reasoning. A key words approach to teaching problem solving prepares students to solve only a very small set of problems both on state assessments and in the real world.

Problem-Solving Steps: A "steps" approach to problem solving gives students a sequenced set of actions to follow to solve a problem. The thinking behind this is aligned to that of teaching a skill like long division—if one follows a set of steps correctly and does the sub-calculations accurately, then one will get a correct final answer to the problem.

Here is an example of “steps” for solving problems found in many instructional materials:

Step 1: Understand the problem.

Step 2: Plan a solution.

Step 3: Solve the problem.

Step 4: Check your work.

The origin of the steps approach to problem solving goes back to 1945 when George Polya, a mathematician at Stanford University, published a book on problem solving called How to Solve It. One the many powerful elements in that book is Polya’s analysis of the phases of the problem-solving process, for which he identified four:

Understanding the problem

Devising a plan

Carrying out the plan

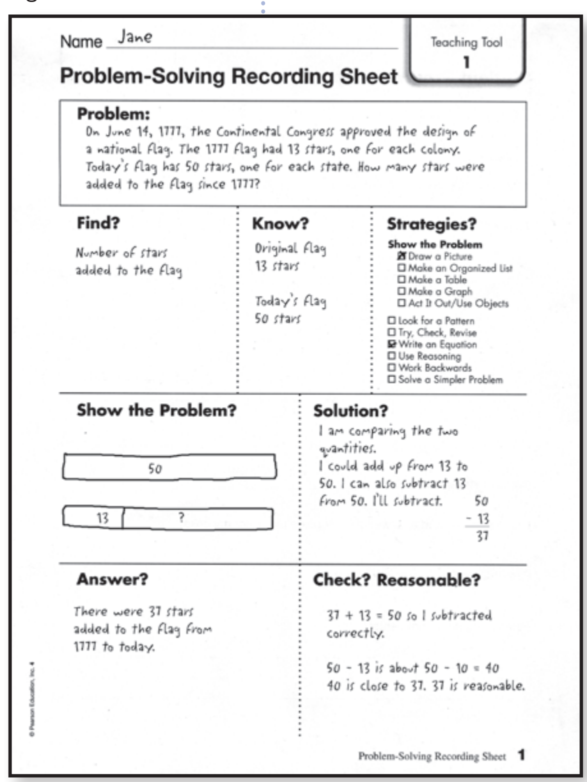
Looking back

Polya’s use of the word phases is intentional and noteworthy for he considered each part or phase of problem solving to be a thoughtful, reflective experience, not a series of steps. Using the word steps promotes at least two misconceptions. First, “steps” suggests (like climbing stairs) that one completes one step and moves off it and onto the next. This is not the way mental processing proceeds for problem solving. One’s understanding of a problem continues to expand and evolve as one works through the phases. Another misconception is that problem solving is like a computational algorithm where there is a sequence of actions to use, which if followed correctly, will lead to the correct solution. Experience shows that problem solving is not an algorithm; there is not a series of steps that guarantee success. Problem solving is a process grounded in sense making and reasoning. Certainly successful problem solvers are skilled at reading and comprehending the words and doing the needed calculations correctly, but problem solving is not a skill.

The message for teachers that should be taken from Polya’s work is that approaching problem solving in a systematic way can be helpful in solving problems but it does not guarantee success. Problem-solving guides based on Polya’s work like that one shown in Figure 1 can be helpful in getting students to think systematically about solving problems, but they should not be presented as “steps for finding the correct answer” for the reasons discussed above.

One reason so many teachers have used a key words or a steps approach to teaching problem solving may be that there are few alternative instructional strategies. But finally, there is now a body of research that provides a new direction for teaching mathematics word problems that will produce success.

Figure 1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Teaching Tool 1

Problem-Solving Recording Sheet

Problem:
On June 14, 1777, the Continental Congress approved the design of a national flag. The 1777 flag had 13 stars, one for each colony. Today's flag has 50 stars, one for each state. How many stars were added to the flag since 1777?

Find?
Number of stars added to the flag

Know?
Original flag 13 stars

Today's flag 50 stars

Strategies?
Show the Problem
- ☒ Draw a Picture
- ☐ Make an Organized List
- ☐ Make a Table
- ☐ Make a Graph
- ☐ Act It Out/Use Objects

- ☐ Look for a Pattern
- ☐ Try, Check, Revise
- ☑ Write an Equation
- ☐ Use Reasoning
- ☐ Work Backwards
- ☐ Solve a Simpler Problem

Show the Problem?

Solution?
I am comparing the two quantities.
I could add up from 13 to 50. I can also subtract 13 from 50. I'll subtract.
50
- 13
37

Answer?
There were 37 stars added to the flag from 1777 to today.

Check? Reasonable?
37 + 13 = 50 so I subtracted correctly.

50 - 13 is about 50 - 10 = 40
40 is close to 37. 37 is reasonable.

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Problem-Solving Recording Sheet 1

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A Visual Approach to Teaching Word Problems

As was noted earlier, problem solving is a process grounded in sense making and reasoning, in particular, quantitative reasoning. Notably, reasoning quantitatively is one of the Standards for Mathematical Practice from the Common Core State Standards. Reasoning quantitatively "entails habits of creating a coherent representation of the problem at hand; considering the units involved; attending to the meaning of the quantities, not just how to compute them; and knowing and flexibly using different properties of operations and objects."[1] In other words, quantitative reasoning involves identifying the quantities in a problem and using reasoning to identify the relationship(s) between or among them.

Here is a word problem whose solution requires quantitative reasoning.

Carrie has 125 U.S. stamps. She has 3 times as many foreign stamps as U.S. stamps. How many stamps does she have altogether?

The quantities in this word problem are:

- the number of U.S. stamps (a known value, 125)
- the number of foreign stamps (an unknown value)
- the total number of foreign and U.S. stamps (an unknown value)

The challenge in solving word problems is not often in identifying and determining the known and unknown quantities. Rather, it is in articulating the relationships between quantities, understanding those relationships, and determining the appropriate operation or operations to show those relationships.

The relationships in this problem are:

- There are 3 times as many foreign stamps as U.S. stamps.
- The total for the number of foreign stamps and the U.S. stamps.

"The challenge for teaching word problems is in helping students use quantitative reasoning."

We know from research that just because a learner can read a word problem, knows all vocabulary in the problem, and can identify the relationships stated in the problem, it does not mean that he or she can find the solution (Knifong & Holton, 1976, 1977). Rather, children who understand operation meanings and can associate relationships between quantities given in word problems with those operation meanings are better problem solvers (see Sowder, 1988). So, the challenge for teaching word problems is in helping students use quantitative reasoning—that is, helping them use reasoning to identify the relationships among the quantities in the problem and connect those relationships to appropriate operations.

Three research findings provide guidance for a way to develop students' quantitative reasoning abilities.

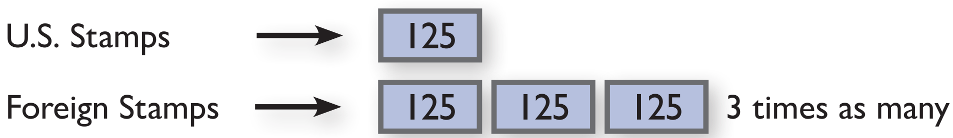
(a) Encourage students to meaningfully represent mathematical word problems. When students begin by representing word problems pictorially, rather than to directly translate the elements of the problems into corresponding mathematical operations, they may more successfully solve these problems and better comprehend the mathematical concepts embedded within them. (Pape, S.J., 2004).

[1] Common Core State Standards, 2010, p. 6

(b) Have students represent problems visually to improve problem-solving performance. Research suggests that having student represent a problem visually results in improved problem-solving performance. (Yancey, Thompson, and Yancey, 1989)

(c) Emphasize to students the importance of the problem structure (i.e., the quantities involved and their relationships) rather than surface features (like key words). (Diezmann and English, 2001, p. 82)

An approach to solving word problems derived from the research findings discussed earlier is to use **bar diagrams** as visual representations to show how quantities in a word problem are related. Seeing those relationships and connecting those to operation meanings helps students to select an appropriate operation for solving the problem. "A diagram can serve to 'unpack' the structure of a problem and lay the foundation for its solution" (Diezmann and English, 2001, p. 77). Nickerson (1994) found that the ability to use diagrams is integral to mathematics thinking and learning.



Here is a bar diagram representing the quantities and their relationships for the word problem given above.

The relationships between the quantities in the problem can be seen in the bar diagram.

- There are 3 times as many foreign stamps as U.S. stamps.

  The set of 3 boxes that each contains 125 shows this relationship. Three equal groups are being joined.

- The total for the number of foreign stamps and the U.S. stamps.

  The combination of all four of the boxes shows this relationship. The total for the three equal groups is being joined with the amount for the one group, but since all groups have the same amount, four equal groups are being joined.

Translating these relationships to numerical expressions requires an understanding of the meanings of the operations. For the first relationship, three quantities are being joined. When quantities are being joined, addition can be used to find the total. When the quantities being joined are equal, multiplication can be used to find the total and is usually more efficient than addition. So, the numerical expression associated with the three boxes representing the number of foreign stamps is 3×125. The numerical expression that shows the joining of the number of foreign stamps and the number of U.S. stamps is $(3 \times 125) + 125$. This expression can be simplified to 4×125; the 4 groups of 125 can easily be seen in the bar diagram. The answer to the problem is that Carrie has 500 stamps altogether.

Figure 2 shows a collection of common one-step word problems; each can be solved using one of the four basic operations. For each, a bar diagram shows the relationship between the quantities. Then one or more equations are given showing the operation or operations that can be used to find the answer. It is important to recognize that a relationship in some word problems can be translated into more than one appropriate equation. For example, Example B shows that how one thinks about the relationship between the quantities in the problem leads to either an addition or subtraction equation; one can add on to 57 to get to 112 or one can subtract 57 from 112.

Figure 2: Bar Diagrams for Addition and Subtraction Situations

Problem Type **Joining**	Example A Total Amount Unknown	Example B Amount Joined Unknown	Example C Initial Amount Unknown
	Kim has 23 antique dolls. Her Father gives her 18 more antique dolls. Now how many antique dolls does she have?	Debbie has saved $57. How much more money does she need in order to have $112?	Tom had some money in his savings account. He then deposited $45 into the same account. Then he had $92 in all. How much did he have in his savings account to start?
Diagram Showing the Relationship	⊢----- ? -----⊣ [23 | 18]	⊢---- 112 ----⊣ [57 | ?]	⊢---- 92 ----⊣ [? | 45]
Description of the Relationship	The two unequal amounts (23 and 18) are known and being joined and the total is unknown.	The initial amount is known (57). The amount being joined to that is unknown. The total is known (112).	The initial amount is unknown. The amount being joined to that is known (45). The total is known (92).
Number Sentence	23 + 18 = ?	57 + ? = 112 112 – 57 = ?	? + 45 = 92 92 – 45 = ?

Problem Type **Separating**	Example D Amount Remaining Unknown	Example E Amount Separated Unknown	Example F Initial Amount Unknown
	Steven has 122 jelly beans. He eats 71 of them in one weekend. How many jelly beans are left?	Carrie has 45 CDs. She gives some to Jo. Now Carrie has 27 left. How many did she give to Jo?	Alan has some marbles. He lost 12 of them. Then he had 32 left. How many did he have before he lost some?
Diagram Showing the Relationship	⊢---- 122 ----⊣ [71 | ?]	⊢---- 45 ----⊣ [? | 27]	⊢---- ? ----⊣ [12 | 32]
Description of the Relationship	The total amount is known (122) and the amount separated from that is known (71). The amount remaining is unknown.	The total amount is known (45) and the amount separated from that is unknown. The amount remaining is known (27).	The total is unknown. The amount separated from the total is known (12) and the amount remaining is known (32).
Number Sentence	122 – 71 = ?	45 – ? = 27 ? + 27 = 45	? – 12 = 32 12 + 32 = ?

Figure 2: (cont'd) Bar Diagrams for Addition and Subtraction Situations

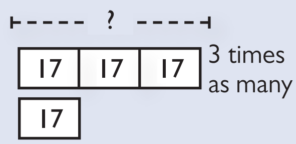
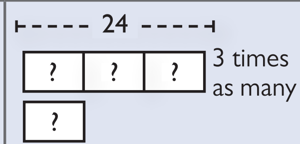
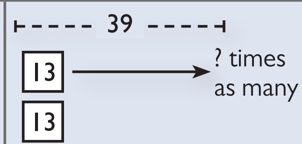
Problem Type **Part-Part-Whole**	Example A Total Amount Unknown	Example B Initial Amount Unknown	Example C Amount Joined Unknown
	Fourteen cats and 16 dogs are in the kennel. How many dogs and cats are in the kennel?	Some adults and 12 children were on a bus. There are 31 people in all on the bus. How many adults were on the bus?	Forty-nine people went on a hike. Six were adults and the rest were children. How many children went on the hike?
Diagram Showing the Relationship	├------ ? ------┤ [14 | 16]	├----- 31 -----┤ [? | 12]	├----- 49 -----┤ [6 | ?]
Description of the Relationship	Each unequal part is known (14 and 16); the whole is unknown.	The first part is unknown, but the second part is known (12). The whole is known (31).	The whole is known (49) and the initial part is known (6). The other part is unknown.
Number Sentence	14 + 16 = ?	? + 12 = 31 31 − 12 = ?	6 + ? = 49 49 − 6 = ?

Problem Type **Comparison**	Example J Amount More (or Less) Unknown	Example K Smaller Amount Unknown	Example L Larger Amount Unknown
	Alex has 47 toy cars. Keisha has 12 cars. How many more cars does Alex have?	Fran spent $84, which was $26 more than Alice spent. How much did Alice spend?	Barney has 23 old coins. Steve has 16 more old coins than Barney. How many old coins does Steve have?
Diagram Showing the Relationship	├----- 47 -----┤ [12 | ?]	├----- 84 -----┤ [? | 26]	├----- ? -----┤ [23 | 16]
Description of the Relationship	Two known amounts (47 and 12) are being compared. The amount more/less is unknown.	The larger amount is known (84), and smaller amount is unknown. The amount more the larger is than the smaller is known (26).	One smaller amount is known (23), and the larger amount is not known. The amount more the larger is than the smaller is known (16).
Number Sentence	47 − 12 = ?	84 − ? = 26 84 − 26 = ?	23 + 16 = ? ? − 23 = 16

Figure 2: (cont'd) Bar Diagrams for Multiplication and Division Situations

Problem Type **Joining Equal Groups**	Example M Total Amount Unknown	Example N Amount per Group Unknown	Example O Number of Groups Unknown
	Kim has 4 photo albums. Each album has 85 pictures. How many photos are in her 4 albums?	Pam had 4 bags and put the same number of apples in each bag. She ended up with 52 apples in bags. How many did she put in each bag?	Fred bought some books that each cost $16. He spent $80 altogether. How many books did he buy?
Diagram Showing the Relationship	? 85 85 85 85	52 ? ? ? ?	80 ? →
Description of the Relationship	Four equal known amounts (85) are being joined to find the unknown total.	A known number (4) of unknown but equal amounts are being joined to give a known total (52).	A known amount (16) is being joined an unknown number of times to itself to get a known total (80).
Number Sentence	$4 \times 85 = ?$	$4 \times ? = 52$ $52 \div 4 = ?$	$? \times 16 = 80$ $80 \div 16 = ?$

Problem Type **Separating Equal Groups**	Example P Amount per Group Unknown	Example Q Number of Groups Unknown	Example R Total Amount Unknown
	Byron has 45 pigeons. He keeps them in 5 pens with the same number of pigeons in each. How many pigeons are in each pen?	A total of 108 children signed up for soccer. How many 18-person teams can be made?	Kim had some cards. She put them into piles of 35 and was able to make 4 piles. How many cards did she have to start?
Diagram Showing the Relationship	45 ? ? ? ? ?	180 18 →	? 35 35 35 35
Description of the Relationship	The total is known (45) and being separated into a known number of equal groups (5) but the amount in each group is unknown.	The total is known (108) and being separated into equal groups of a known amount (18). The number of equal groups needed to match the total is unknown.	The total amount is unknown. It is separated into a known number of groups (4) with a known equal amount in each (35).
Number Sentence	$45 \div 5 = ?$	$108 \div 18 = ?$ $18 \times ? = 108$	$? \div 4 = 35$ $4 \times 35 = ?$

Problem Type **Comparison**	Example S Larger Amount Unknown	Example T Smaller Amount Unknown	Example U Number of Times as Many Unknown
	Alex has 17 toy cars. Keisha has 3 times as many. How many cars does Keisha have?	Barney has 24 old coins. This is 3 times more coins than Steve has. How many old coins does Steve have?	Ann's teacher is 39 years old. Ann is 13 years old. Ann's teacher is how many times as old as Ann?
Diagram Showing the Relationship			
Description of the Relationship	The smaller amount is known (17) and the larger amount is a given number of times more (3). The larger quantity is not known.	The larger amount is known (24) and is a given number of times greater than the small amount (3). The smaller amount is not known.	The larger amount (39) and the smaller amount (13) are known. How many times more the larger amount is than the smaller amount is not known.
Number Sentence	$3 \times 17 = ?$	$3 \times ? = 24$ $24 \div 3 = ?$	$? \times 13 = 39$ $39 \div 13 = ?$

For multiple-step problems such as the one cited earlier involving Carrie's stamps, multiple bar diagrams are used to help answer the hidden question (i.e., sub-problem) and then answer the final question. In the Carrie problem, the hidden question was to find the total number of foreign stamps; the answer to that was then used to answer the question stated in the problem.

One of the powerful attributes of these bar diagrams is that all show relationships between parts and wholes. This coherence in visual representations helps students see not only the connections between the diagrams but also connections between and among operations. An important part of understanding operations is to know all relationships between and among the four operations.

Suggestions for Teaching

Here are a few suggestions for how bar diagrams can be an integral part of teaching and learning mathematics (Diezmann, & English, 2001).

- Model bar diagrams on a regular basis; not just in special lessons but frequently when word problems are encountered.
- Discuss the structure of bar diagrams and connect them to quantities in the word problem and to operation meanings.
- Use bar diagrams to focus on the structure of a word problem, not surface features like key words.
- Encourage students to use bar diagrams to help them understand and solve problems.

The mathematics community has consistently promoted the use of visual representations to foster students' mathematical reasoning (NCTM, 2000). The Standards for Mathematical Practice in the Common Core State Standards for Mathematics are the latest reminders of the importance of helping students "see" mathematical concepts in order to understand the structure of a problem situation. Bar diagrams provide that visual structure to make mathematics accessible to a much wider group of students. They are a powerful tool for developing visual literacy and success with problem solving for ALL students.

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